

Algebraic method for pseudo-Hermitian Hamiltonians

Jun-Qing Li¹, Yan-Gang Miao^{1,2,*} and Zhao Xue¹

¹*School of Physics, Nankai University, Tianjin 300071,
People's Republic of China*

²*Bethe Center for Theoretical Physics and Institute of Physics, University of Bonn,
Nussallee 12, D-53115 Bonn, Germany*

Abstract

An algebraic method for pseudo-Hermitian Hamiltonian systems is proposed through introducing the operator η_+ and the η_+ -pseudo-Hermitian adjoint of states and then redefining annihilation and creation operators to be η_+ -pseudo-Hermitian (not Hermitian) adjoint of each other. As an example, a parity-pseudo-Hermitian Hamiltonian is constructed and analyzed in detail. Its real spectrum is obtained by means of the algebraic method, where the corresponding operator η_+ is found to be PV through a specific choice of V . The operator V is given in such a way that on the one hand this P -pseudo-Hermitian Hamiltonian is also PV -pseudo-Hermitian self-adjoint and on the other hand PV ensures a real spectrum and a positive-definite inner product. Moreover, when the P -pseudo-Hermitian system is extended to the canonical noncommutative space with noncommutative spatial coordinates and noncommutative momenta as well, the first order noncommutative correction of energy levels is calculated, and in particular the reality of energy spectra and the positive-definiteness of inner products are found to be not altered by the noncommutativity.

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*Corresponding author. E-mail address: miaoyg@nankai.edu.cn

1 Introduction

A non-Hermitian Hamiltonian with a complex potential usually corresponds to complex eigenvalues and such a system does not maintain the conservation of probability. However, the non-Hermitian Hamiltonian with a class of quasi-Hermiticity was proposed [1] in which the real eigenvalues and the conservation of probability are possible. Recently the eigenvalues and eigenstates of a non-Hermitian Hamiltonian associated with some symmetry have been paid more attention to. Interestingly, if the η -pseudo-Hermitian self-adjoint condition [2] is satisfied,

$$H = \eta^{-1} H^\dagger \eta, \quad (1)$$

where the invertible operator η is linear Hermitian and the Hamiltonian H is diagonalizable, this Hamiltonian has a set of biorthonormal basis [3]. The Hamiltonian with such a symmetry is called an η -pseudo-Hermitian Hamiltonian [2, 4, 5, 6]. Therefore, one can deal with the pseudo-Hermitian system in terms of the biorthonormal basis. In recent years the pseudo-Hermiticity has been investigated extensively. For instance, in the aspect related to the PT -symmetric quantum mechanics [7], the PT -symmetric Hamiltonian was shown [4, 5, 6] to be equivalent to the parity-pseudo-Hermitian one. In addition, some experiments [8, 9] on PT -symmetric (parity-pseudo-Hermitian) Hamiltonians have been carried out in the region of optics.

Roughly speaking, there exist two methods which are used to study an ordinary (Hermitian) Hamiltonian system. The usual one focuses on solving the Schrödinger equation under certain boundary conditions in order to have eigenvalues and eigenstates. The associated calculations are complicated sometimes. The other method, which is useful for dealing with the systems such as the harmonic oscillator, is associated closely with annihilation and creation operators and their algebraic relations, and therefore it may be called an algebraic method. Nevertheless, in the quantum mechanics with non-Hermitian Hamiltonians, the former method is commonly adopted in literature, see, for instance, the review article [10], while the latter, i.e. the algebraic method cannot be utilized directly because some parameters in a non-Hermitian Hamiltonian are complex. Such a complexification of parameters gives rise to the result that the annihilation and creation operators of the non-Hermitian Hamiltonian are no longer Hermitian adjoint (conjugate) of each other. Fortunately, a non-Hermitian Hamiltonian is usually connected with a certain symmetry of pseudo-Hermiticity with which one can apply the algebraic method to deal with the non-Hermiticity. In this paper we give a general proposal of algebraic methods for pseudo-Hermitian Hamiltonian systems, and as an application we construct a parity-pseudo-Hermitian Hamiltonian and analyze its spectrum and inner product, and further extend this system to the canonical noncommutative space. We shall show that our way can in principle be applied to an arbitrary η -pseudo-Hermitian system.

This paper is organized as follows. In the next section, we propose our algebraic method for an arbitrary η -pseudo-Hermitian system by postulating the existence of η_+ operator¹ (see also footnote 5 in section 2 for more details), by defining the generalized inner product with respect to η_+ , and then by redefining annihilation and creation operators which are η_+ -pseudo-Hermitian adjoint of each other. We shall see that the key step of this method is to give a suitable η_+ which not only makes the original η -pseudo-Hermitian self-adjoint be raised to the η_+ -pseudo-Hermitian self-adjoint but also leads to a real spectrum with lower boundedness and a positive-definite inner product. In section 3, as an application of our proposal, we construct a parity-pseudo-Hermitian Hamiltonian and give a specific choice of $\eta_+ = PV$ through introducing the operator V . Then we redefine the annihilation and creation operators in such a way that they are PV -pseudo-Hermitian adjoint of each other in this pseudo-Hermitian system. We obtain a real energy spectrum and a positive-definite inner product. In addition, we can verify that our P -pseudo-Hermitian Hamiltonian is also PV -pseudo-Hermitian self-adjoint, which provides the consistency of choosing PV as η_+ . In section 4, the P -pseudo-Hermitian Hamiltonian is extended to the canonical noncommutative phase space with noncommutative coordinates and momenta as well. We calculate the noncommutative correction of energy levels up to the first order of noncommutative parameters, and in particular we work out an interesting result that the reality of energy spectra and the positive-definiteness of inner products are not altered by the noncommutativity of phase space. Finally, we make a brief conclusion in section 5.

2 Algebraic method for η_+ -pseudo-Hermitian systems

Let us now give our proposal of the algebraic method for an arbitrary η -pseudo-Hermitian system under the prerequisite that this system will also be η_+ -pseudo-Hermitian self-adjoint (see footnote 1). The key step of this method is to find out the operator η_+ which can give positive-definite inner products [1]. We note that the proposal is quite general.

At first, the condition that an η_+ -pseudo-Hermitian observable \mathcal{A} should obey takes the following form [2, 6, 11], i.e. the η_+ -pseudo-Hermitian self-adjoint condition,

$$\mathcal{A} = \mathcal{A}^\ddagger \equiv \eta_+^{-1} \mathcal{A}^\dagger \eta_+, \quad (2)$$

where the subscript “+” means that η_+ associates with a positive-definite inner product in the pseudo-Hermitian system, and the superscript “ \ddagger ” stands for the η_+ -pseudo-Hermitian adjoint of an operator. The η_+ -pseudo-Hermitian adjoint of a state is defined by

$${}^\ddagger\langle\varphi(x)| \equiv \langle\varphi(x)|\eta_+, \quad (3)$$

¹In general, η_+ does not coincide with η . In our model η is just the parity operator but η_+ is set to be PV where V is obtained quite nontrivially. For the concrete procedure, see eqs. (25) and (26).

where $\langle\varphi(x)|$ denotes the usual Hermitian adjoint of the state $|\varphi(x)\rangle$, $\langle\varphi(x)| = (|\varphi(x)\rangle)^\dagger$, and then the modified inner product in the Hilbert space for an η_+ -pseudo Hermitian Hamiltonian system has the form,

$${}^\ddagger\langle\varphi(x)|\psi(x)\rangle = \langle\varphi(x)|\eta_+|\psi(x)\rangle, \quad (4)$$

which can be understood as a generalized inner product. Since η was called by Pauli [2] the indefinite metric in the Hilbert space, η_+ is then called the positive-definite metric because it gives rise to [1, 12] a real and positive-definite norm or probability, $\langle\psi(x)|\eta_+|\psi(x)\rangle$. Note that the operator η_+ is in general required [2, 4, 5, 6] to be linear Hermitian² and invertible, which ensures not only the reality of the average of physical observables but also the reality and positivity of the probability. In addition, we emphasize the consistency of the self-adjoint condition and the generalized inner product. That is to say, the self-adjoint condition (eq. (2)) guarantees that the average of \mathcal{A} is real [2] with respect to the modified inner product (eq. (4)), and simultaneously it implies that \mathcal{A} is self-adjoint with respect to this modified inner product, i.e., ${}^\ddagger\langle\mathcal{A}\varphi(x)|\psi(x)\rangle = {}^\ddagger\langle\varphi(x)|\mathcal{A}\psi(x)\rangle$. We point out that it is the requirement of positive norms that makes it a quite nontrivial task to find out a suitable η_+ for an η -pseudo-Hermitian Hamiltonian, see the two sections below for a detailed analysis.

Next, we define the new creation operator (which is different from that of the Hermitian quantum mechanics) as the η_+ -pseudo-Hermitian adjoint of the annihilation operator as follows:

$$a^\ddagger \equiv \eta_+^{-1} a^\dagger \eta_+. \quad (5)$$

Note that the new creation and annihilation operators are η_+ -pseudo-Hermitian adjoint of each other, that is, we have³ $a = (a^\ddagger)^\ddagger$. In addition, we can prove that a and a^\ddagger are η_+ -pseudo-Hermitian adjoint of each other with respect to the generalized inner product eq. (4), that is, considering eqs. (4) and (5) we get

$$\begin{aligned} {}^\ddagger\langle a\varphi(x)|\psi(x)\rangle &= \langle\varphi(x)|a^\ddagger\eta_+|\psi(x)\rangle = \langle\varphi(x)|\eta_+ (\eta_+^{-1} a^\dagger \eta_+) |\psi(x)\rangle = \langle\varphi(x)|\eta_+ a^\dagger |\psi(x)\rangle \\ &= {}^\ddagger\langle\varphi(x)|a^\ddagger\psi(x)\rangle. \end{aligned} \quad (6)$$

This shows that the generalization of the creation operator is consistent with that of the inner product. The formula $a = (a^\ddagger)^\ddagger$ reduces to the one we are quite familiar with, i.e. $a = (a^\dagger)^\dagger$, when η_+ takes the identity operator, i.e. when an η_+ -pseudo-Hermitian system turns out to be a Hermitian one. Considering the well-known commutation relations satisfied by the usual annihilation and creation operators in the conventional quantum mechanics, we require that the newly defined annihilation and creation operators in the η_+ -pseudo-Hermitian quantum

²In our recent work [12] we discuss the anti-linear anti-Hermitian case and obtain some interesting results.

³It is easy to verify this equality, that is, $(a^\ddagger)^\ddagger = \eta_+^{-1} (\eta_+^{-1} a^\dagger \eta_+)^\dagger \eta_+ = \eta_+^{-1} \eta_+ a \eta_+^{-1} \eta_+ = a$, where the Hermiticity of η_+ has been utilized.

mechanics comply with

$$[a, a^\dagger] = 1, \quad [a, a] = 0 = [a^\dagger, a^\dagger], \quad (7)$$

which reduces consistently to the usual commutation relations when η_+ becomes the identity.

At last, we define the corresponding number operator in the pseudo-Hermitian quantum mechanics as follows:

$$N \equiv a^\dagger a, \quad (8)$$

which, as a physical observable, is of course η_+ -pseudo-Hermitian self-adjoint⁴, i.e. $N^\dagger = N$. We can also prove that the number operator N is simultaneously self-adjoint with respect to the generalized inner product (eq. (4)), i.e.,

$$\begin{aligned} {}^\dagger\langle N\varphi(x)|\psi(x)\rangle &= \langle\varphi(x)|N^\dagger\eta_+|\psi(x)\rangle = \langle\varphi(x)|\eta_+(\eta_+^{-1}N^\dagger\eta_+)|\psi(x)\rangle = \langle\varphi(x)|\eta_+N^\dagger|\psi(x)\rangle \\ &= {}^\dagger\langle\varphi(x)|N\psi(x)\rangle. \end{aligned} \quad (9)$$

Using eqs. (7) and (8), we can verify the following commutation relations in the pseudo-Hermitian quantum mechanics,

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a. \quad (10)$$

As a consequence, we establish the algebraic method for an η_+ -pseudo-Hermitian system. The remaining task is just to deduce some useful formulas, such as the n -particle state and the ladder property of creation and annihilation operators. Incidentally, we mention that the above proposal reduces consistently to that of the ordinary (Hermitian) quantum mechanics when η_+ becomes the identity operator.

Let us derive the n -particle state. If $|0\rangle$ stands for the ground state and a annihilates the ground state, $a|0\rangle = 0$, we calculate the average of $a^n(a^\dagger)^n$ with respect to the ground state and its η_+ -pseudo-Hermitian adjoint (see eq. (3)) by repeatedly using eq. (7),

$$\begin{aligned} & {}^\dagger\langle 0|a^n(a^\dagger)^n|0\rangle \\ &= {}^\dagger\langle 0|a^{n-1}a^\dagger a(a^\dagger)^{n-1}|0\rangle + {}^\dagger\langle 0|a^{n-1}(a^\dagger)^{n-1}|0\rangle \\ &\vdots \\ &= {}^\dagger\langle 0|a^{n-1}(a^\dagger)^n a|0\rangle + n {}^\dagger\langle 0|a^{n-1}(a^\dagger)^{n-1}|0\rangle \\ &= n {}^\dagger\langle 0|a^{n-1}(a^\dagger)^{n-1}|0\rangle \\ &= n(n-1) {}^\dagger\langle 0|a^{n-2}(a^\dagger)^{n-2}|0\rangle \\ &\vdots \\ &= n! {}^\dagger\langle 0|0\rangle. \end{aligned} \quad (11)$$

⁴Considering eqs. (5) and (8), we have $N^\dagger = (\eta_+^{-1}a^\dagger\eta_+a)^\dagger = \eta_+^{-1}(\eta_+^{-1}a^\dagger\eta_+a)^\dagger\eta_+ = \eta_+^{-1}a^\dagger\eta_+a\eta_+^{-1}\eta_+ = a^\dagger a = N$, where the Hermiticity of η_+ has been used. In general, the number operator is non-Hermitian for an η_+ -pseudo-Hermitian system, i.e. $N^\dagger \neq N$. We can check that on the one hand $N^\dagger = \eta_+ N \eta_+^{-1} \neq N$ because of $[N, \eta_+] \neq 0$, and on the other hand $\langle N\varphi(x)|\psi(x)\rangle = \langle\varphi(x)|\eta_+ N \eta_+^{-1}|\psi(x)\rangle \neq \langle\varphi(x)|N\psi(x)\rangle$. See the following two sections for the details.

If $|n\rangle$ is defined by

$$|n\rangle \equiv \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \quad (12)$$

we obtain its η_+ -pseudo-Hermitian adjoint by using eqs. (3) and (5),

$${}^\dagger\langle n| = \langle 0|\frac{1}{\sqrt{n!}}((a^\dagger)^n)^\dagger\eta_+ = \langle 0|\frac{1}{\sqrt{n!}}\eta_+a^n = {}^\dagger\langle 0|\frac{1}{\sqrt{n!}}a^n. \quad (13)$$

Therefore, we can rewrite eq. (11) as

$${}^\dagger\langle n|n\rangle = {}^\dagger\langle 0|\frac{1}{n!}a^n(a^\dagger)^n|0\rangle = {}^\dagger\langle 0|0\rangle. \quad (14)$$

Moreover, by using eqs. (7), (8) and (12) and considering $a|0\rangle = 0$ again, we derive

$$\begin{aligned} N|n\rangle &= \frac{1}{\sqrt{n!}}a^\dagger a(a^\dagger)^n|0\rangle \\ &= \frac{1}{\sqrt{n!}}(a^\dagger)^2a(a^\dagger)^{n-1}|0\rangle + \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \\ &\quad \vdots \\ &= \frac{1}{\sqrt{n!}}(a^\dagger)^{n+1}a|0\rangle + n\frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \\ &= n|n\rangle. \end{aligned} \quad (15)$$

Combining eq. (14) and eq. (15), we obtain

$${}^\dagger\langle n|N|n\rangle = n{}^\dagger\langle n|n\rangle = n{}^\dagger\langle 0|0\rangle. \quad (16)$$

Consequently, if the ground state is normalized⁵ with respect to the generalized inner product (eq. (4)), i.e. ${}^\dagger\langle 0|0\rangle = 1$, the state defined by eq. (12) is convinced to be the expected n -particle state and the operator N defined by eq. (8) is then confirmed to be the desired number operator.

Now we calculate the ladder property of creation and annihilation operators. It is straightforward from eq. (12) that we have

$$a^\dagger|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^{n+1}|0\rangle = \frac{\sqrt{n+1}}{\sqrt{(n+1)!}}(a^\dagger)^{n+1}|0\rangle = \sqrt{n+1}|n+1\rangle.$$

⁵As claimed above, it is crucial to keep the generalized inner product (eq. (4)) positive definite for an η -pseudo-Hermitian system. This is equivalent to maintaining the norm of the ground state positive definite, see eq. (14). This requirement leads us to look for a suitable η_+ which not only has the property ${}^\dagger\langle 0|0\rangle = 1$ but also makes the η -pseudo-Hermitian system η_+ -pseudo-Hermitian self-adjoint. In the next two sections the exact forms of η_+ operators will be provided for our models. We mention that η is just the parity operator P in our models and thus cannot be separated to the product of an operator O and its Hermitian adjoint O^\dagger . In such a case, the positiveness of $\langle\psi(x)|P|\psi(x)\rangle$ is not ensured. That is the reason why it is crucial for us to find out a positive-definite metric η_+ . However, if η is separable, i.e. $\eta = OO^\dagger$, the norm $\langle\psi(x)|OO^\dagger|\psi(x)\rangle$ is positive definite (see refs. [4, 5, 6] for illustrations). In this case, it is not necessary to introduce η_+ .

Multiplying the above equation by the operator a from the left and using eqs. (7), (8) and (15), we get

$$aa^\dagger|n\rangle = (a^\dagger a + 1)|n\rangle = (N + 1)|n\rangle = (n + 1)|n\rangle.$$

Combining the above two equations, we obtain $a|n + 1\rangle = \sqrt{n + 1}|n\rangle$. As a result, we give the following ladder properties for the operators a^\dagger and a , respectively,

$$a^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle, \quad a|n\rangle = \sqrt{n}|n - 1\rangle, \quad (17)$$

which indeed shows that a^\dagger has the function of creation and a that of annihilation as expected.

In addition, we emphasize that the unitarity of time evolution is guaranteed with respect to the modified inner product (eq. (4)) in the η_+ -pseudo-Hermitian quantum mechanics. Considering the η_+ -pseudo-Hermitian self-adjoint of the Hamiltonian, i.e. $H = \eta_+^{-1}H^\dagger\eta_+$, and the time evolution of an initial state $|\psi(0)\rangle$, $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$, we have

$$\begin{aligned} {}^\dagger\langle\psi(t)|\psi(t)\rangle &\equiv \langle\psi(t)|\eta_+|\psi(t)\rangle = \langle\psi(0)|e^{+iH^\dagger t}\eta_+e^{-iHt}|\psi(0)\rangle \\ &= \langle\psi(0)|\eta_+(\eta_+^{-1}e^{+iH^\dagger t}\eta_+)e^{-iHt}|\psi(0)\rangle = \langle\psi(0)|\eta_+(e^{+iHt})e^{-iHt}|\psi(0)\rangle, \\ &= \langle\psi(0)|\eta_+|\psi(0)\rangle \equiv {}^\dagger\langle\psi(0)|\psi(0)\rangle, \end{aligned} \quad (18)$$

which gives the unitary time evolution.

As a summary, we point out that the characteristic of our algebraic method is not to introduce the biorthonormal basis but only to adopt the orthonormal basis of Hamiltonians. Although the biorthonormal basis has extensively been applied [4, 5, 6] to pseudo-Hermitian Hamiltonian systems, it is interesting to investigate whether the usual treatment (to consider just the orthonormal basis of Hamiltonians but not that of the Hermitian conjugate of Hamiltonians) is still available to such non-Hermitian systems. Our way to realize this goal is to find out a suitable operator η_+ and then to redefine the annihilation and creation operators that are adjoint of each other with respect to the generalized inner product (eq. (4)). For the concrete procedure, see the next two sections. We emphasize that the operator η_+ is in general nontrivial, that is, it is impossible to reduce η_+ to be the identity through a basis transformation because the Hamiltonian of an η_+ -pseudo-Hermitian system is no longer Hermitian (self-adjoint) with respect to the usual inner product: $\langle H\varphi(x)|\psi(x)\rangle \neq \langle\varphi(x)|H\psi(x)\rangle$, but η_+ -pseudo-Hermitian self-adjoint with respect to the generalized inner product (see eqs. (3) and (4)): ${}^\dagger\langle H\varphi(x)|\psi(x)\rangle = {}^\dagger\langle\varphi(x)|H\psi(x)\rangle$.

3 Parity-pseudo-Hermitian system

In this section we investigate a concrete non-Hermitian Hamiltonian with the parity-pseudo-Hermiticity by means of the algebraic method provided in the above section. We add two

non-Hermitian terms which are proportional to $i(x_1 + x_2)$ and $i(p_1 + p_2)$, respectively, to the Hamiltonian of an isotropic planar oscillator, and then give a new Hamiltonian:

$$H = \frac{1}{2} (p_1^2 + x_1^2) + \frac{1}{2} (p_2^2 + x_2^2) + i[A(x_1 + x_2) + B(p_1 + p_2)], \quad (19)$$

where A and B are real parameters; x_j and p_j , $j = 1, 2$, are two pairs of canonical coordinates and their conjugate momenta, they are all Hermitian and satisfy the standard Heisenberg commutation relations:

$$[x_j, p_k] = i\delta_{jk}, \quad [x_j, x_k] = 0 = [p_j, p_k], \quad j, k = 1, 2, \quad (20)$$

where \hbar is set to be unity. Eq. (19) describes a decoupled two-dimensional oscillator. This Hamiltonian seems to be too simple⁶, and normally it is enough for us to analyze its one-dimensional part in this section. However, this is a good enough model for us to illustrate our algebraic method. In particular, the decoupled oscillator will turn out to be a coupled (nontrivial) two-dimensional oscillator when it is extended to the noncommutative space in the next section. That is the reason why we choose such a decoupled two-dimensional oscillator in this section.

The Hamiltonian eq. (19) is obviously non-Hermitian, $H \neq H^\dagger$, but it possesses the parity-pseudo-Hermiticity,

$$H = P^{-1}H^\dagger P, \quad (21)$$

where P is the parity operator which is linear Hermitian and invertible. Furthermore, we notice that the Hamiltonian can easily be diagonalized and rewritten as

$$H = H_1 + H_2 + (A^2 + B^2), \quad (22)$$

where the new variables are defined by

$$\begin{aligned} H_1 &\equiv \frac{1}{2}(P_1^2 + X_1^2), & H_2 &\equiv \frac{1}{2}(P_2^2 + X_2^2), \\ P_1 &\equiv p_1 + iB, & X_1 &\equiv x_1 + iA, \\ P_2 &\equiv p_2 + iB, & X_2 &\equiv x_2 + iA. \end{aligned} \quad (23)$$

Eq. (22), together with eq. (23), looks like the usual Hamiltonian of a decoupled two-dimensional harmonic oscillator, but in fact, it is not the case because X_j and P_j are non-Hermitian, $X_j \neq X_j^\dagger$ and $P_j \neq P_j^\dagger$, although they satisfy the same commutation relations as eq. (20),

$$[X_j, P_k] = i\delta_{jk}, \quad [X_j, X_k] = 0 = [P_j, P_k], \quad j, k = 1, 2. \quad (24)$$

⁶Several highly nontrivial models have been studied in our work [12, 13] by means of the algebraic method proposed in the present paper.

Now we begin the investigation of the P -pseudo-Hermitian system governed by the Hamiltonian eq. (19) or eq. (22). We shall see that the key point is to give a suitable η_+ for the system.

After making numerous attempts and tedious calculations, we find out a two-step way for the construction of the desired operator η_+ . In the first step, inspired by Lee and Wick [14], we define operator V as follows,

$$V = (-1)^{H_1+H_2-1}, \quad (25)$$

which is invertible. Note that here V is P -pseudo-Hermitian self-adjoint, i.e., $V = P^{-1}V^\dagger P$, which is different from the case in ref. [14] where a Hermitian operator was introduced. Moreover, we point out that V is defined in terms of the Hamiltonian, which is more intuitive than the definition of the corresponding operator C [7] in the PT -symmetric quantum mechanics, see our related work [13] for the details. Then, in the second step we set η_+ to be the product of P and V ,

$$\eta_+ = PV, \quad (26)$$

which is linear Hermitian and invertible. Note that V is linear non-Hermitian because H_1 and H_2 are non-Hermitian. It is easy to prove the Hermiticity of η_+ by considering the P -pseudo-Hermitian self-adjoint of V ($V = P^{-1}V^\dagger P$), that is, $\eta_+^\dagger = V^\dagger P^\dagger = P(P^{-1}V^\dagger P) = PV = \eta_+$. In addition, because η_+ is related to the Hamiltonian through V the generalized inner product with respect to this η_+ (see eq. (4)) can be called a dynamical inner product as the CPT inner product [7] was called.

Using the above η_+ we now define the operator a_j^\dagger as the PV -pseudo-Hermitian adjoint of the operator a_j , i.e. $a_j^\dagger \equiv (PV)^{-1}a_j^\dagger(PV)$. Considering eqs. (23)-(26) we obtain

$$a_j^\dagger = \frac{1}{\sqrt{2}}(X_j - iP_j), \quad j = 1, 2, \quad (27)$$

if a_j takes the form,

$$a_j = \frac{1}{\sqrt{2}}(X_j + iP_j), \quad j = 1, 2, \quad (28)$$

and further derive their algebraic relations,

$$[a_j, a_k^\dagger] = \delta_{jk}, \quad [a_j, a_k] = 0 = [a_j^\dagger, a_k^\dagger], \quad j, k = 1, 2. \quad (29)$$

In the following, we shall show that a_j^\dagger and a_j are indeed the creation and annihilation operators we are searching for by confirming the positiveness of the inner product with respect to PV .

In accordance with the proposal given in the above section, we can now write the number operator⁷ which is PV -pseudo-Hermitian self-adjoint,

$$N_j = a_j^\dagger a_j, \quad j = 1, 2, \quad (30)$$

⁷Repeated subscripts do not sum except for extra indications.

and get the expected commutation relations by using eqs. (29) and (30),

$$[N_j, a_k^\dagger] = a_j^\dagger \delta_{jk}, \quad [N_j, a_k] = -a_j \delta_{jk}, \quad j, k = 1, 2. \quad (31)$$

Furthermore, given $|n_j\rangle$ a set of eigenstates of the number operator N_j ,

$$N_j |n_j\rangle = n_j |n_j\rangle, \quad j = 1, 2, \quad (32)$$

if its inner product defined by eq. (4) is positive definite, a_j^\dagger and a_j can finally be convinced to be the creation and annihilation operators that satisfy the property of ladder operators,

$$a_j^\dagger |n_j\rangle = \sqrt{n_j + 1} |n_j + 1\rangle, \quad a_j |n_j\rangle = \sqrt{n_j} |n_j - 1\rangle, \quad j = 1, 2. \quad (33)$$

At present we verify that the generalized inner product defined by eq. (4) is positive definite for our choice $\eta_+ = PV$. Due to eq. (16), we only need to prove the normalization of the ground state, $\langle 0|PV|0\rangle = 1$. Utilizing eqs. (25), (27), (28), (30) and (32), we have $V|0\rangle = (-1)^{H_1+H_2-1}|0\rangle = (-1)^{N_1+N_2}|0\rangle = |0\rangle$, and thus obtain $\langle 0|PV|0\rangle = \langle 0|P|0\rangle$. Furthermore, considering the wavefunction of the ground state, $\varphi_0(X_j) = \frac{1}{\sqrt[4]{\pi}} \exp(-\frac{1}{2}X_j^2 + BX_j)$, where $j = 1, 2$, and X_j 's are the coordinates whose operators are defined by eq. (23), we can calculate the generalized inner product with respect to the ground state in terms of the Cauchy's residue theorem of the complex function theory (see Figure 1 for the details),

$$\begin{aligned} \langle 0|PV|0\rangle &= \langle 0|P|0\rangle = \int_{-\infty+iA}^{+\infty+iA} \overline{\varphi}_0(X_j) P \varphi_0(X_j) dX_j \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty+iA}^{+\infty+iA} \exp[-(x_j - iA)^2] d(x_j + iA) \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-x_j^2) dx_j \\ &= 1, \quad j = 1, 2, \end{aligned} \quad (34)$$

where $\overline{\varphi}_0$ denotes the complex conjugate of φ_0 . Note that the symbols X_j and x_j in the above equation no longer stand for operators but coordinates. Eq. (34) definitely gives the normalization of the ground state, which, together with eq. (16), leads to $\langle n_j|PV|n_j\rangle = 1$, where $j = 1, 2$. That is, we at last prove the positive definiteness of the generalized inner product (defined by eq. (4)) for the set of eigenstates of the number operator (given by eq. (30)).

Alternatively, by using Mathematica we can exactly solve the wavefunctions for any excited states related with the Hamiltonian eq. (19) or eq. (22):

$$\varphi_{n_1 n_2}(X_1, X_2) = \varphi_{n_1}(X_1) \varphi_{n_2}(X_2), \quad (35)$$

where

$$\varphi_{n_j}(X_j) = \frac{1}{\sqrt[4]{\pi}} (2^{n_j} n_j!)^{-\frac{1}{2}} e^{-\frac{1}{2}X_j^2 + BX_j} H_{n_j}(X_j), \quad j = 1, 2, \quad (36)$$

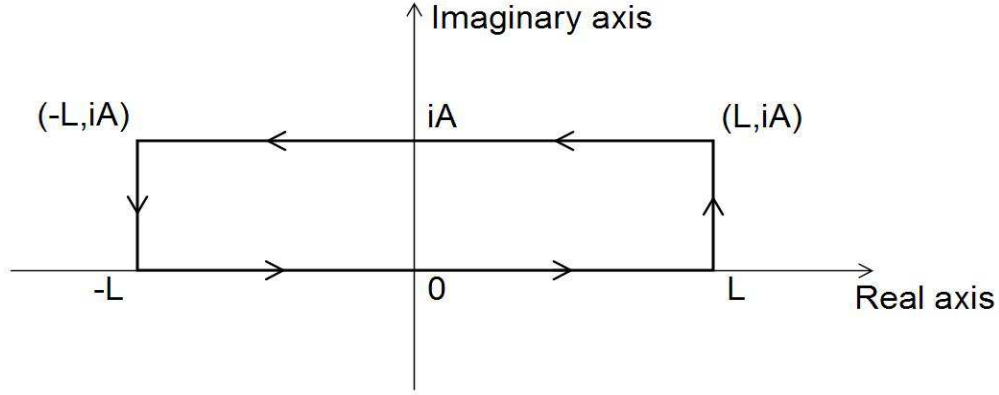


Figure 1: We choose the rectangle with length L and width A in the complex plane as the contour. Inside the rectangle the integrand $\bar{\varphi}_0(x+iy) P \varphi_0(x+iy)$ is analytic, i.e. no singular points exist, thus the contour integration is zero by means of the Cauchy's residue theorem in a simply connected domain. The integrations along the two perpendicular (right and left) sides equal $\exp(-L^2) \int_0^A \exp(y^2) [-\sin(2Ly) \pm i \cos(2Ly)] dy$, they are also vanishing when the length L tends to infinity. In consequence, the integrations on the top and bottom sides of the rectangle equal if along the same direction and in the limit $L \rightarrow \infty$, as given explicitly by eq. (34).

and $H_{n_j}(X_j)$ denotes the Hermite polynomial of the n_j -th degree with the argument X_j whose operator is given in eq. (23). Therefore, by considering $V \varphi_{n_j}(X_j) = (-1)^{n_j} \varphi_{n_j}(X_j)$ and using the same contour as in Figure 1 we can prove that the inner product for oscillator #1 or oscillator #2 is orthogonal and normalized, i.e.,

$$\langle n_j | PV | m_j \rangle = \int_{-\infty+iA}^{+\infty+iA} \bar{\varphi}_{n_j}(X_j) PV \varphi_{m_j}(X_j) dX_j = \delta_{n_j m_j}, \quad j = 1, 2. \quad (37)$$

This shows from an alternative point of view that the positive definiteness of the inner product is guaranteed.

As a result, using eqs. (27), (28) and (30) we can easily rewrite the Hamiltonian eq. (22) in terms of the number operators as follows:

$$H = (N_1 + N_2 + 1) + (A^2 + B^2), \quad (38)$$

and then give its real and positive spectrum,

$$E_{n_1 n_2} = (n_1 + n_2 + 1) + (A^2 + B^2), \quad n_1, n_2 = 0, 1, 2, \dots \quad (39)$$

In an alternative way, we can get the same spectrum eq. (39) if we let the Hamiltonian eq. (38) act on the eigenfunction eq. (35).

At the end of this section we point out that the system with the original P -pseudo-Hermitian symmetry also has the PV -pseudo-Hermiticity, i.e. $H = (PV)^{-1}H^\dagger(PV)$. This property is quite obvious when we verify it by using eqs. (22)-(25), that is, $(PV)^{-1}H^\dagger(PV) = V^{-1}(P^{-1}H^\dagger P)V = V^{-1}HV = H$, where $[H, V] = 0$ is used in the last equality. This shows that the PV -pseudo-Hermiticity is a consistent symmetry (the prerequisite is met, see the beginning of section 2) for the non-Hermitian quantum system governed by one of the Hamiltonian formulations eqs. (19), (22) and (38). We conclude that this PV -pseudo-Hermitian system possesses a real spectrum with lower boundedness and a positive-definite inner product and thus it is an acceptable quantum theory.

4 Noncommutative extension of the system

In the 1930s, Heisenberg [15] proposed a kind of lattice structures of spacetimes, i.e. the quantized spacetime now called the noncommutative spacetime, in order to overcome the ultraviolet divergence in quantum field theory. Later Snyder [16] applied the idea of spacetime noncommutativity to construct the Lorentz invariant field theory with a small length scale cut-off. Since the Seiberg-Witten's seminal work [17] on describing some low-energy effective theory of open strings by means of a noncommutative gauge theory, the physics founded on noncommutative spacetimes has been studied intensively, see, for instance, some review articles [18]. As a result, it is quite natural to ask how an η_+ -pseudo-Hermitian Hamiltonian behaves on a noncommutative space. That is, it is interesting to investigate whether the η_+ -pseudo-Hermitian symmetry, real spectrum and positive-definite inner product of the system remain or not when the pseudo-Hermitian system is generalized to a noncommutative space. Incidentally, one of the authors of the present paper established [19] a noncommutative theory of chiral bosons and found that the self-duality that exists in the usual chiral bosons is broken in the noncommutative chiral bosons.

We consider a general two-dimensional canonical noncommutative space with noncommutative spatial coordinates and noncommutative momenta as well,

$$[\hat{x}_j, \hat{x}_k] = i\theta\epsilon_{jk}, \quad [\hat{p}_j, \hat{p}_k] = i\tilde{\theta}\epsilon_{jk}, \quad [\hat{x}_j, \hat{p}_k] = i\delta_{jk}, \quad j, k = 1, 2, \quad (40)$$

where $\epsilon_{12} = -\epsilon_{21} = 1$, and θ and $\tilde{\theta}$ independent of coordinates and momenta are real noncommutative parameters which are much smaller than the Planck constant. Therefore, we generalize our system (eq. (19)) to this noncommutative space in a straightforward way,

$$\hat{H} = \frac{1}{2}(\hat{p}_1^2 + \hat{x}_1^2) + \frac{1}{2}(\hat{p}_2^2 + \hat{x}_2^2) + i[A(\hat{x}_1 + \hat{x}_2) + B(\hat{p}_1 + \hat{p}_2)]. \quad (41)$$

In accordance with the commutation relations in the two spaces, i.e. eqs. (20) and (40), we establish the following relationship between the commutative and noncommutative spaces

up to the first order of θ and $\tilde{\theta}$,

$$\hat{x}_j = x_j - \frac{1}{2}\theta\epsilon_{jk}p_k, \quad \hat{p}_j = p_j + \frac{1}{2}\tilde{\theta}\epsilon_{jk}x_k, \quad (\text{summation for repeated subscripts}), \quad (42)$$

and then rewrite eq. (41) in terms of the coordinates and momenta of the commutative space still up to the first order of θ and $\tilde{\theta}$,

$$\begin{aligned} \mathcal{H} = & \frac{1}{2}(p_1^2 + x_1^2) + \frac{1}{2}(p_2^2 + x_2^2) + i[A(x_1 + x_2) + B(p_1 + p_2)] \\ & + \frac{1}{2}(\theta + \tilde{\theta})(x_2p_1 - x_1p_2) - i\left[\frac{1}{2}B\tilde{\theta}(x_1 - x_2) - \frac{1}{2}A\theta(p_1 - p_2)\right]. \end{aligned} \quad (43)$$

The last two terms in the above Hamiltonian give the noncommutative corrections, where the first that presents the coupling of the two one-dimensional oscillators is Hermitian while the second is not. Note that the original P -pseudo-Hermiticity obviously remains in the noncommutative extension, i.e. $\hat{H} = P^{-1}\hat{H}^\dagger P$ or $\mathcal{H} = P^{-1}\mathcal{H}^\dagger P$, which can easily be seen from eq. (41) or eq. (43). However, we point out that eq. (41) is symmetric under the permutation of dimension #1 and dimension #2 while eq. (43) does not possess such a permutation symmetry because the relationship between the commutative and noncommutative spaces (eq. (42)) breaks this symmetry under the first order approximation to the noncommutative parameters.

After carefully analyzing eq. (43) and making a lot of attempts we introduce new variables as follows:

$$\begin{aligned} \mathcal{P}_1 &\equiv p_1 + i\mathcal{B}_1, & \mathcal{X}_1 &\equiv x_1 + i\mathcal{A}_1, \\ \mathcal{P}_2 &\equiv p_2 + i\mathcal{B}_2, & \mathcal{X}_2 &\equiv x_2 + i\mathcal{A}_2, \end{aligned} \quad (44)$$

where new real parameters \mathcal{A}_j and \mathcal{B}_j , $j = 1, 2$, are defined by

$$\begin{aligned} \mathcal{A}_1 &\equiv A + \frac{1}{2}B\theta, & \mathcal{A}_2 &\equiv A - \frac{1}{2}B\theta, \\ \mathcal{B}_1 &\equiv B - \frac{1}{2}A\tilde{\theta}, & \mathcal{B}_2 &\equiv B + \frac{1}{2}A\tilde{\theta}. \end{aligned} \quad (45)$$

The new variables are non-Hermitian like that in the commutative case (see eq. (23)) but they satisfy the same commutation relations as eq. (20) or eq. (24),

$$[\mathcal{X}_j, \mathcal{P}_k] = i\delta_{jk}, \quad [\mathcal{X}_j, \mathcal{X}_k] = 0 = [\mathcal{P}_j, \mathcal{P}_k], \quad j, k = 1, 2, \quad (46)$$

which is very important in the noncommutative extension. Therefore, we can realize the partial diagonalization of eq. (43) in terms of the new variables up to the first order of θ and $\tilde{\theta}$,

$$\mathcal{H} = \frac{1}{2}(\mathcal{P}_1^2 + \mathcal{X}_1^2) + \frac{1}{2}(\mathcal{P}_2^2 + \mathcal{X}_2^2) + \frac{1}{2}(\theta + \tilde{\theta})(\mathcal{X}_2\mathcal{P}_1 - \mathcal{X}_1\mathcal{P}_2) + (A^2 + B^2). \quad (47)$$

We note that the third term in eq. (47) gives the first order correction to the noncommutative parameters. This term describes the coupling of oscillator #1 and oscillator #2 and thus needs to be dealt with particularly.

Following the procedure stated in the above section for searching for the operator V and furthermore considering in particular the coupling term appeared in eq. (47), we at first find out the corresponding operator \mathcal{V} for the noncommutative case,

$$\mathcal{V} = (-1)^{\mathcal{H}_1 + \mathcal{H}_2 - 1}, \quad (48)$$

where \mathcal{H}_j 's are defined as

$$\mathcal{H}_j \equiv \frac{1}{2}(\mathcal{P}_j^2 + \mathcal{X}_j^2), \quad j = 1, 2. \quad (49)$$

Note that \mathcal{V} is linear and invertible but non-Hermitian, and it is also P -pseudo-Hermitian self-adjoint as V (see eq. (25)), i.e. using eqs. (46), (48) and (49) we have $P^{-1}\mathcal{V}^\dagger P = P^{-1}(-1)^{\mathcal{H}_1 + \mathcal{H}_2 - 1}P = (-1)^{\mathcal{H}_1 + \mathcal{H}_2 - 1} = \mathcal{V}$. Then we give the expected operator η_+ as follows,

$$\eta_+ = P\mathcal{V}, \quad (50)$$

which can be proved to be Hermitian though \mathcal{V} is not. Considering the P -pseudo-Hermiticity of \mathcal{V} , we have $\eta_+^\dagger = \mathcal{V}^\dagger P^\dagger = P(P^{-1}\mathcal{V}^\dagger P) = P\mathcal{V} = \eta_+$.

We can now define \mathbf{a}_j^\dagger as the $P\mathcal{V}$ -pseudo-Hermitian adjoint of \mathbf{a}_j : $\mathbf{a}_j^\dagger \equiv (P\mathcal{V})^{-1}\mathbf{a}_j^\dagger(P\mathcal{V})$. We emphasize that it is a difficult task to set a suitable \mathbf{a}_j and then to determine its $P\mathcal{V}$ -pseudo-Hermitian adjoint because the coupling term appears in the noncommutative case (see eq. (47)). After making numerous attempts we at last work out \mathbf{a}_j ,

$$\mathbf{a}_j = \frac{1}{2} \left((\eta_{jk} + i\epsilon_{jk})\mathcal{X}_k + (i\eta_{jk} - \epsilon_{jk})\mathcal{P}_k \right), \quad (\text{summation for repeated subscripts}), \quad (51)$$

where $\eta_{jk} \equiv \text{diag}(1, -1)$, and therefore obtain its $P\mathcal{V}$ -pseudo-Hermitian adjoint by considering eqs. (46), (48) and (49),

$$\mathbf{a}_j^\dagger = \frac{1}{2} \left((\eta_{jk} - i\epsilon_{jk})\mathcal{X}_k - (i\eta_{jk} + \epsilon_{jk})\mathcal{P}_k \right), \quad (\text{summation for repeated subscripts}). \quad (52)$$

We can show that the algebraic relations of \mathbf{a}_j and \mathbf{a}_j^\dagger are same as eq. (29), which is our desired result. Further, we give the number operator which is $P\mathcal{V}$ -pseudo-Hermitian self-adjoint,

$$\mathcal{N}_j = \mathbf{a}_j^\dagger \mathbf{a}_j, \quad j = 1, 2, \quad (53)$$

and find that \mathcal{N}_j , \mathbf{a}_j and \mathbf{a}_j^\dagger have the same commutation relations as eq. (31). Similarly, for a given set of eigenstates of the number operator \mathcal{N}_j , i.e. $\mathcal{N}_j|n_j\rangle = n_j|n_j\rangle$, we can prove (see

below) that \mathbf{a}_j^\dagger and \mathbf{a}_j are indeed the creation and annihilation operators we are looking for, that is, they satisfy the property of ladder operators eq. (33).

Now we can write the Hamiltonian eq. (47) in a completely diagonalized form by means of the number operator \mathcal{N}_i ,

$$\mathcal{H} = (\mathcal{N}_1 + \mathcal{N}_2 + 1) + \frac{1}{2}(\theta + \tilde{\theta})(\mathcal{N}_1 - \mathcal{N}_2) + (A^2 + B^2), \quad (54)$$

and easily give the real and positive energy spectrum up to the first order of the noncommutative parameters,

$$\mathcal{E}_{n_1 n_2} = (n_1 + n_2 + 1) + \frac{1}{2}(\theta + \tilde{\theta})(n_1 - n_2) + (A^2 + B^2), \quad n_1, n_2 = 0, 1, 2, \dots \quad (55)$$

Note that the first order correction of the spectrum is proportional to the difference between the eigenvalue of oscillator #1 and that of oscillator #2. We point out that the first order correction of the energy spectrum is vanishing when the noncommutative parameters satisfy the special relation $\theta + \tilde{\theta} = 0$, in which case higher order corrections might be considered. Moreover, if $\theta + \tilde{\theta} \neq 0$ but $n_1 - n_2 = 0$, i.e. the energy eigenvalues of oscillator #1 and oscillator #2 equal, there is no first order correction for the spectrum, either. For instance, it is obvious that the energy level of the ground state is not modified because of $n_1 = n_2 = 0$. However, we emphasize that the noncommutative corrections of the eigenfunction are non-vanishing even for the two special cases because the eigenfunction, as stated in the above section, has the same formulation (see the next paragraph for a detailed analysis) as eqs. (35) and (36) with the replacement of X_j by the new coordinates \mathcal{X}_j ($j = 1, 2$) given in eq. (44) and thus contains the noncommutative parameter θ through \mathcal{X}_j . This would be seen more evidently from eq. (54) which is the diagonalized form of eq. (47).

We turn to the proof of the positive-definite inner product in the noncommutative case, which shows as in the commutative case that \mathbf{a}_j^\dagger and \mathbf{a}_j are the creation and annihilation operators that satisfy the property of ladder operators eq. (33). Because the coupling term is commutative with the Hamiltonian of oscillator #1 and oscillator #2 in eq. (47), that is⁸,

$$\left[\mathcal{X}_2 \mathcal{P}_1 - \mathcal{X}_1 \mathcal{P}_2, \frac{1}{2}(\mathcal{P}_1^2 + \mathcal{X}_1^2) + \frac{1}{2}(\mathcal{P}_2^2 + \mathcal{X}_2^2) \right] = 0, \quad (56)$$

we conclude that the eigenfunction of the total Hamiltonian (eq. (47)) is the product of the eigenfunctions of the oscillator #1 Hamiltonian and oscillator #2 Hamiltonian. As a result, it takes the same form as that obtained in the above section just with the replacement of X_j by \mathcal{X}_j ($j = 1, 2$) given in eq. (44). For example, the eigenfunction of the ground state for one of the oscillators is: $\varphi_0(\mathcal{X}_j) = \frac{1}{\sqrt[4]{\pi}} \exp(-\frac{1}{2}\mathcal{X}_j^2 + \mathcal{B}_j \mathcal{X}_j)$, where $j = 1, 2$, and repeated subscripts do not sum. Similar to the commutative case in section 3 (see eq. (34)), by

⁸Such a commutativity can be seen more clearly from eq. (54), i.e. $[\mathcal{N}_1 - \mathcal{N}_2, \mathcal{N}_1 + \mathcal{N}_2 + 1] = 0$.

using $\mathcal{V}|0\rangle = (-1)^{\mathcal{H}_1+\mathcal{H}_2-1}|0\rangle = (-1)^{\mathcal{N}_1+\mathcal{N}_2}|0\rangle = |0\rangle$, we have $\langle 0|P\mathcal{V}|0\rangle = \langle 0|P|0\rangle$, and can therefore prove the normalization of the generalized inner product of the ground state in terms of the Cauchy's residue theorem together with the contour chosen in Figure 1, i.e. $\langle 0|P\mathcal{V}|0\rangle = \langle 0|P|0\rangle = 1$. Moreover, we can also prove the orthogonality and normalization of the inner products of excited states, like eq. (37) for the noncommutative case. This completes the proof of the positive definiteness of the generalized inner product defined by eq. (4) with $\eta_+ = P\mathcal{V}$.

As analyzed in section 3 for the commutative case, we can verify straightforwardly from eq. (54) that the Hamiltonian possesses the $P\mathcal{V}$ -pseudo-Hermiticity in the noncommutative case, i.e. $\mathcal{H} = (P\mathcal{V})^{-1}\mathcal{H}^\dagger(P\mathcal{V})$, which shows that the construction of operator \mathcal{V} is consistent with the original P -pseudo-Hermiticity. As a consequence, in the noncommutative generalization we find that the reality of energy spectra with lower boundedness and the positive definiteness of inner products maintain if we choose $P\mathcal{V}$ as the pseudo-Hermitian symmetry for the system depicted by the Hamiltonian eq. (43), eq. (47), or eq. (54). The reason relies on the existence of the η_+ -pseudo-Hermiticity without which such properties may not be maintained in both the commutative and noncommutative cases. Incidentally, it is quite evident that the eigenvalues and eigenfunctions of our noncommutative generalization reduce to their commutative counterparts (see section 3) when the parameters θ and $\tilde{\theta}$ tend to zero. This shows that our noncommutative extension is consistent.

5 Conclusion

In this paper, we provide a general algebraic method for an arbitrary η -pseudo-Hermitian quantum system and note that the crucial point of this method is to find out a suitable η_+ (positive-definite metric) which corresponds to a real spectrum and a positive-definite inner product. The pseudo-Hermitian system with such properties can then be accepted quantum mechanically. We apply our method to a P -pseudo-Hermitian system and then extend it to the canonical noncommutative space with both noncommutative spatial coordinates and noncommutative momenta. For the two systems, we find out the specific η_+ operators and prove the reality of energy spectra and the positive definiteness of inner products, and moreover, to the latter system we obtain the first order correction of spectra to the noncommutative parameters. Here we have to mention an earlier work [20] which also dealt with a non-Hermitian Hamiltonian system. Although the real spectrum was given there, the non-Hermiticity of Hamiltonian was not properly treated and more severely the positive-definiteness of inner products was completely ignored. In fact, the annihilation and creation operators defined in ref. [20] are no longer Hermitian adjoint of each other due to the non-Hermiticity of Hamiltonian, which gives rise to the problems pointed out above. We have solved the problems in terms of our proposal of the algebraic method. In addition, we

emphasize that it is not inevitable to adopt the biorthonormal basis for pseudo-Hermitian systems, and that it is enough to use the usual orthonormal basis with a suitable choice of the operator η_+ and a corresponding redefinition of annihilation and creation operators. At last we just mention that we have applied the algebraic method to the fourth-order derivative Pais-Uhlenbeck oscillator model [21] that can be brought within the framework of pseudo-Hermitian systems, and in particular have found [12] a new phenomenon — the spontaneous breaking of permutation symmetry in this model.

As a further consideration, it is interesting to investigate the conservation of probability in the noncommutative generalizations of pseudo-Hermitian systems. The conservation of probability remains obviously in our noncommutative generalization of the P -pseudo-Hermitian system, see eqs. (18) and (43). However, a general pseudo-Hermitian system on noncommutative spacetimes usually contains complicated potentials and thus its conservation of probability is not so obvious as in our model. Related problems are being studied and results will be given separately.

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